A Calculus for Stochastic Interventions: Causal Effect Identification and Surrogate Experiments

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Outline
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- Hard/atomic interventions vs. Soft/non-atomic interventions
- Graphical representation
- Inferences rules for soft interventions ($\sigma$-calculus)
- Imperfect surrogate experiments
- Conclusions
Motivating example
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W & \rightarrow (\text{previous GPA}) \\
Z & \rightarrow (\text{motivation})
\end{align*}
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- Motivation depends (among other not observed factors) on the previous GPA.
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\[ \mathcal{G} \text{ Natural (current) Regime} \]
• Using machine learning, and with enough data, a student's GPA can be predicted with small error given other features i.e., $P(y | w, z, x)$. 

\[ G \text{ Natural (current) Regime} \]
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• This distribution is a model that reflects the current/natural regime, but we are interested in taking decisions to improve the students GPA.
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Taking decisions amount to intervening the current regime. Hence, we are interested in predicting student’s GPA receiving tutoring in a hypothetical (unrealized) reality.
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This is a causal inference question!
Some types of Interventions
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• **Stochastic:** \( \sigma_X = P*(x|w) \) sets the variable \( X \) to follow a given probability distribution conditional on a set of variables \( W \).
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  - Students with low GPA enter a raffle for 80% of the spots, other interested students enter for the remaining 20%.
Hard/Atomic Interventions
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• What if we make tutoring mandatory for every student?
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\[ W \]
\[ \text{(previous GPA)} \]

\[ Z \]
\[ \text{(motivation)} \]

\[ X \]
\[ \text{(tutoring)} \]

\[ Y \]
\[ \text{(GPA)} \]
Hard/Atomic Interventions

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\[ (\text{previous GPA}) \]

\[ W \]

\[ (\text{motivation}) \]

\[ Z \]

\[ (\text{tutoring}) \]

\[ \mathcal{G} \text{ Natural (current) Regime} \]

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\[ \text{(GPA)} \]
Hard/Atomic Interventions

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\begin{align*}
\text{(previous GPA)} & \quad W \\
\text{(motivation)} & \quad Z \\
\text{(tutoring)} & \quad X \\
\text{(GPA)} & \quad Y \\
\end{align*}
\]

\[\mathcal{G}\text{ Natural (current) Regime}\]

\[\mathcal{G}_X\text{ Intervened (hypothesized) Regime}\]

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Hard/Atomic Interventions

• What if we make tutoring mandatory for every student?

\[ \begin{align*}
X &\quad \text{(tutoring)} \\
Z &\quad \text{(motivation)} \\
W &\quad \text{(previous GPA)} \\
Y &\quad \text{(GPA)}
\end{align*} \]

\[ \text{Intervention } do(X = 1) \]

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\mathcal{G} &\quad \text{Natural (current) Regime} \\
\mathcal{G}_X &\quad \text{Intervened (hypothesized) Regime}
\end{align*} \]

Instead of \( P(y \mid X=1) \) we are reasoning about \( P(y \mid do(X=1)) \), or, more generally, \( P(y; \sigma_X=do(X=1)) \).
Soft Interventions
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• A more realistic, or interesting, type of intervention is, for example, to consider the effect of making tutoring mandatory for students with historically low GPA and only to them, on their current GPA.
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\[ \text{Intervention } \sigma_X = 1[W = 1] \]

Assign tutoring only to students with low GPA.

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σ-calculus (simplified)
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- Insertion/deletion of observations:

\[
P(y \mid w, t; \sigma_X) = P(y \mid w; \sigma_X) \quad \text{if } (Y \perp T \mid W) \text{ in } \mathcal{G}_{\sigma_X}
\]
σ-calculus (simplified)

- **Insertion/deletion of observations:**

\[ P(y \mid w, t; \sigma_X) = P(y \mid w; \sigma_X) \]  
if \((Y \perp T \mid W)\) in \(\mathcal{G}_{\sigma_X}\)

- **Change of regimes under observation:**

\[ P(y \mid x, w; \sigma_X) = P(y \mid x, w) \]  
if \((Y \perp Z \mid W)\) in \(\mathcal{G}_{\sigma_X^X}\) and \(\mathcal{G}_X\)
σ-calculus (simplified)

• Insertion/deletion of observations:

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• Change of regimes under observation:

\[ P(y \mid x, w; \sigma_X) = P(y \mid x, w) \quad \text{if} \quad (Y \perp Z \mid W) \text{ in } G_{\sigma_X \bar{X}} \text{ and } G_{\bar{X}} \]

• Change of regimes without observations:

\[ P(y \mid w; \sigma_X) = P(y \mid w) \quad \text{if} \quad (Y \perp Z \mid W) \text{ in } G_{\sigma_X \bar{X}(\bar{W})} \text{ and } G_{\bar{X}(\bar{W})} \]
Using $\sigma$-calculus

$$P(y; \sigma_x) = \sum_{w, z} P(y \mid x, w, z; \sigma_x) P(x \mid w, z; \sigma_x) P(w, z; \sigma_x)$$
Using \( \sigma \)-calculus

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**Rule 1** \((X \perp Z \mid W)\) in \(\mathcal{G}_{\sigma_X}\)
Using $\sigma$-calculus

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Rule 2 \( (Y \perp X \mid W, Z) \) in $G_{\sigma_X X}$ and $G_{X}$
Using $\sigma$-calculus

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P(y; \sigma_X) = \sum_{w, z} P(y \mid x, w, z; \sigma_X) P(x \mid w, z; \sigma_X) P(w, z; \sigma_X)
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**Rule 2** \( (Y \perp X \mid W, Z) \) in \( \mathcal{G}_{\sigma_X}X \) and \( \mathcal{G}_X \)
Using $\sigma$-calculus

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Rule 1 \((X \perp Z | W)\) in $\mathcal{G}_{\sigma_X}$

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Rule 2 \((Y \perp X | W, Z)\) in $\mathcal{G}_{\sigma_XX}$ and $\mathcal{G}_{X}$

\[ = \sum_{w,z} P(y | x, w, z)P(x | w; \sigma_X)P(w, z) \]

Rule 3 \((W, Z \perp X)\) in $\mathcal{G}_{\sigma_XX}$ and $\mathcal{G}_{X}$
Using $\sigma$-calculus

\[ P(y; \sigma_X) = \sum_{w,z} P(y \mid x, w, z; \sigma_X)P(x \mid w, z; \sigma_X)P(w, z; \sigma_X) \]

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Rule 2 $(Y \perp X \mid W, Z)$ in $\mathcal{G}_{\sigma_X \bar{X}}$ and $\mathcal{G}_{\bar{X}}$

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Defined by $\sigma_X$
Using $\sigma$-calculus

$$P(y; \sigma_X) = \sum_{w,z} P(y | x, w, z; \sigma_X)P(x | w, z; \sigma_X)P(w, z; \sigma_X)$$

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Rule 2 $(Y \perp X | W, Z)$ in $\mathcal{G}_{\sigma_XY}$ and $\mathcal{G}_X$

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Estimable from current regime

Defined by $\sigma_X$
Surrogate Experiments
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• Experiments over a set of surrogate variables $Z$ may be more accessible to manipulation than the target effect $\sigma_X$, e.g., randomizing diet vs randomizing cholesterol.
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- Experiments over a set of surrogate variables $Z$ may be more accessible to manipulation than the target effect $\sigma_X$, e.g., randomizing diet vs randomizing cholesterol.

- Those surrogate experiments can be leveraged to identify the effect of the interventions of interest.
Surrogate Experiments
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- **Input:** \( \{P(v), P(v | \sigma Z = P^*(Z|X))\} \)
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- **Input:** \( \{P(v), P(v \mid \sigma_z=P^*(Z|X))\} \)
- **Query:** \( P(y \mid \sigma_x=P^*(X|R)) \)
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\[
P(y; \sigma_X)
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- **Query:** \( P(y | \sigma_X = P^*(X|R)) \)
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P(y; \sigma_X) = \sum_{r, w, x, z} P(r)P(x | r; \sigma_X)P(z | r, x, w)P(w | r) \sum_{x'} P(y | r, x', z; \sigma_Z)P(x' | r)
\]
Surrogate Experiments

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- **Query:** $P(y | \sigma_X=P*(X|R))$
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From surrogate experiment
Surrogate Experiments

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**Natural regime**

**From surrogate experiment**
Surrogate Experiments

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- **Query:** \( P(y \mid \sigma_X = P^*(X \mid R)) \)
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P(y; \sigma_X) = \sum_{r, w, x, z} P(r)P(x \mid r; \sigma_X)P(z \mid r, x, w)P(w \mid r) \sum_{x'} P(y \mid r, x', z; \sigma_Z)P(x' \mid r)
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Defined by intervention

Natural regime

From surrogate experiment
Summary of the Results
We introduce a set of inference rules called \( \sigma \)-calculus, which generalizes Pearl’s do-calculus, to reason about the effect of general types of interventions. Further, we provide a syntactical method for deriving and verifying claims about such interventions given a causal graph.
Summary of the Results

We introduce a set of inference rules called $\sigma$-calculus, which generalizes Pearl’s do-calculus, to reason about the effect of general types of interventions. Further, we provide a syntactical method for deriving and verifying claims about such interventions given a causal graph.

We develop an efficient procedure to determine the identifiability of the (conditional) effect of non-atomic interventions from a combination of observational and experimental data given a causal diagram.
Proposed Strategy
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1. Encode qualitative assumptions natural and intervened domain graphically.
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Diagrams annotated with $\sigma_X$ nodes.
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4. Construct an estimator from the available data.
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Diagrams annotated with $\sigma_X$ nodes.

Use $\sigma$-calculus or equivalent algorithmic procedure.
Conclusions
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• These rules can be used to identify the effect of interventions from a combination of observational and experimental data.
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• These rules can be used to identify the effect of interventions from a combination of observational and experimental data.

• Our algorithm searches for a reduction of the effect of interest to the set of observed distributions (observational and experimental); if found, it returns a corresponding mapping expression.
Thank you!